

SEMICLASSICAL LIMITS OF ORE EXTENSIONS AND A POISSON GENERALIZED WEYL ALGEBRA

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ABSTRACT. We observe [8, Proposition 4.1] that Poisson polynomial extensions appear as semiclassical limits of a class of Ore extensions. As an application, a Poisson generalized Weyl algebra A_1 considered as a Poisson version of the quantum generalized Weyl algebra is constructed and its Poisson structures are studied. In particular, it is obtained a necessary and sufficient condition such that A_1 is Poisson simple and established that the Poisson endomorphisms of A_1 are Poisson analogues of the endomorphisms of the quantum generalized Weyl algebra.

1. INTRODUCTION

Let h be a nonzero, nonunit, non-zero-divisor and central element of an algebra R such that R/hR is commutative. Then R/hR becomes a Poisson algebra with Poisson bracket

$$\{\bar{a}, \bar{b}\} = \overline{h^{-1}(ab - ba)}$$

for all $\bar{a}, \bar{b} \in R/hR$. In such case, R/hR is called a semiclassical limit of R . Since the Poisson bracket of R/hR is induced by the commutation rule of R , it is expected that the Poisson structures of R/hR are heavily related to the algebraic structures of deformation algebras of R since they are induced by the same algebra R . In fact, algebraic structures of quantized algebras are analogues to Poisson structures of their semiclassical limits as seen in many cases [3], [4], [6], [9], [11] and [12]. A main aim of this paper is to give a method how to construct Poisson algebras considered as a Poisson version of algebras related to a class of Ore extensions and an example illustrating this method. Namely, we observe [8, Proposition 4.1] that Poisson polynomial extensions appear as semiclassical limits of a class of Ore extensions, construct a Poisson generalized Weyl algebra A_1 as an application and we verify that the Poisson endomorphisms of A_1 are Poisson analogues of the endomorphisms of quantum generalized Weyl algebra.

The Poisson polynomial extensions were constructed as a Poisson version of Ore extensions by the second author in [10] and the quantum generalized Weyl algebra $A(a(h), q)$ over a Laurent polynomial ring in one variable was constructed by Bavula [1] and the endomorphisms of $A(a(h), q)$ were completely classified by Kitchin and Launois [7] in the case when $a(h)$ is not invertible and q is not a root of unity. Here we find a natural map Γ from Ore extensions onto their semiclassical limits and then, as an application, we construct a Poisson generalized Weyl algebra A_1 induced from $A(a(h), q)$ by using Γ . Next we find a necessary and sufficient condition such that A_1 is Poisson simple and establish that the Poisson endomorphisms of A_1 are Poisson analogues of the endomorphisms of $A(a(h), q)$ by classifying the Poisson endomorphisms of A_1 .

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Assume throughout the paper that all algebras have unity and that the base field is the complex number field \mathbb{C} .

Let us begin with recalling the following basic terminologies.

Definition 1.1. (1) Let \mathbb{F} be a commutative \mathbb{C} -algebra. Given an \mathbb{F} -automorphism α on an \mathbb{F} -algebra R , an \mathbb{F} -linear map δ is said to be a *left α -derivation* on R if

$$\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$$

for all $a, b \in R$. For such pair (α, δ) , there exists a skew polynomial \mathbb{F} -algebra (or Ore extension) $R[z; \alpha, \delta]$. Refer to [5, Chapter 2] for details of the skew polynomial ring.

(2) A commutative \mathbb{C} -algebra A is said to be a *Poisson algebra* if there exists a bilinear product $\{-, -\}$ on A , called a *Poisson bracket*, such that $(A, \{-, -\})$ is a Lie algebra and $\{ab, c\} = a\{b, c\} + \{a, c\}b$ for all $a, b, c \in A$.

We recall [10, 1.1]. A derivation α on A is said to be a *Poisson derivation* if

$$\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\}$$

for all $a, b \in A$. Let α be a Poisson derivation and let δ be a derivation on A . If the pair (α, δ) satisfies the following condition

$$(1.1) \quad \delta(\{a, b\}) - \{\delta(a), b\} - \{a, \delta(b)\} = \alpha(a)\delta(b) - \delta(a)\alpha(b)$$

for all $a, b \in A$, then the commutative polynomial algebra $A[z]$ becomes a Poisson algebra with Poisson bracket

$$\{z, a\} = \alpha(a)z + \delta(a)$$

for all $a \in A$. Such Poisson algebra $A[z]$ is called a Poisson polynomial extension (or Poisson Ore extension) and denoted by $A[z; \alpha, \delta]_p$. (In [10, 1.1], $\{z, a\}$ is defined by $\{z, a\} = -\alpha(a)z - \delta(a)$ for all $a \in A$.) If $\alpha = 0$ then we write $A[z; \delta]_p$ for $A[z; 0, \delta]_p$ and if $\delta = 0$ then we write $A[z; \alpha]_p$ for $A[z; \alpha, 0]_p$.

(3) An ideal I of a Poisson algebra A is said to be a *Poisson ideal* if $\{I, A\} \subseteq I$. A Poisson ideal P is said to be *Poisson prime* if, for all Poisson ideals I and J , $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. If A is noetherian then a Poisson prime ideal of A is a prime ideal by [3, Lemma 1.1(d)].

2. POLYNOMIAL EXTENSIONS

Let t be an indeterminate.

Notation 2.1. Let a 5-tuple $(\mathbf{K}, \mathbb{F}, A, t-1, (\alpha, \delta))$ satisfy the following conditions (1)-(5):

(1) Assume that \mathbf{K} is an infinite subset of the set $\mathbb{C} \setminus \{0, 1\}$.

(2) Assume that \mathbb{F} is a subring of the ring of regular functions on $\mathbf{K} \cup \{1\}$ containing $\mathbb{C}[t, t^{-1}]$. That is,

$$(2.1) \quad \mathbb{C}[t, t^{-1}] \subseteq \mathbb{F} \subseteq \{f/g \in \mathbb{C}(t) \mid f, g \in \mathbb{C}[t] \text{ such that } g(1) \neq 0, g(\lambda) \neq 0 \forall \lambda \in \mathbf{K}\}.$$

(3) Assume that A is an \mathbb{F} -algebra generated by x_1, \dots, x_n . Note that A is also a \mathbb{C} -algebra since $\mathbb{C} \subseteq \mathbb{F}$.

(4) Assume that $t-1$ is a nonzero, nonunit and non-zero-divisor of A such that the factor $A_1 = A/(t-1)A$ is commutative. (Note that $t-1 \in \mathbb{F}$ is a central element of A and thus $(t-1)A$ is an ideal of A .)

(5) Assume that α and δ are \mathbb{F} -linear maps from A into itself such that α is an automorphism, δ is a left α -derivation and the pair (α, δ) satisfies the condition

$$(2.2) \quad (\alpha - \text{id})(A) \subseteq (t - 1)A, \quad \delta(A) \subseteq (t - 1)A,$$

where id is the identity map on A .

By Notation 2.1(5), there exists the skew polynomial \mathbb{F} -algebra

$$B := A[z; \alpha, \delta].$$

For each $\lambda \in \mathbf{K} \cup \{1\}$, $(t - \lambda)A$ and $(t - \lambda)B$ are ideals of A and B respectively since $t - \lambda$ is a central element of A and B . Set

$$A_\lambda := A/(t - \lambda)A, \quad B_\lambda := B/(t - \lambda)B.$$

For an element a of A or B , denote by \bar{a} the canonical image of a in A_λ and B_λ . For each $\lambda \in \mathbf{K}$, define

$$\begin{aligned} \alpha_\lambda : A_\lambda &\longrightarrow A_\lambda, & \alpha_\lambda(\bar{a}) &= \overline{\alpha(a)}, \\ \delta_\lambda : A_\lambda &\longrightarrow A_\lambda, & \delta_\lambda(\bar{a}) &= \overline{\delta(a)}. \end{aligned}$$

Lemma 2.2. (1) For each $\lambda \in \mathbf{K}$, α_λ is an automorphism and δ_λ is a left α_λ -derivation.

(2) For each $\lambda \in \mathbf{K}$, $B_\lambda \cong A_\lambda[z; \alpha_\lambda, \delta_\lambda]$ as \mathbb{C} -algebras.

Proof. (1) Since α and δ are \mathbb{F} -linear maps and $t - \lambda \in \mathbb{F}$, α_λ and δ_λ are well-defined. Since α is an automorphism on A , α_λ is an automorphism on A_λ . Moreover δ_λ is a left α_λ -derivation because δ is a left α -derivation.

(2) Let $\psi : A \longrightarrow A_\lambda[z; \alpha_\lambda, \delta_\lambda]$ be the map defined by $\psi(a) = \bar{a}$. Then, by [5, Proposition 2.4], there is an extension $\bar{\psi}$ of ψ to B such that $\bar{\psi}(z) = z$ since $z\bar{a} = \overline{\alpha(a)}z + \overline{\delta(a)}$ for all $a \in A$. Clearly $\ker \bar{\psi} = (t - \lambda)B$ and $\bar{\psi}$ is surjective. \square

Since $t - 1$ is a nonzero, nonunit and non-zero-divisor of A , it is also a nonzero, nonunit and non-zero-divisor of B . Moreover B_1 is a commutative \mathbb{C} -algebra by (2.2). Hence A_1 and B_1 are Poisson algebras with Poisson brackets

$$(2.3) \quad \{\bar{a}, \bar{b}\} = \overline{(t - 1)^{-1}(ab - ba)}$$

for all $a, b \in A$ and $a, b \in B$ by [2, III.5.4]. The \mathbb{C} -algebras A_1 and B_1 are said to be *semiclassical limits* of A and B respectively. For each $\lambda \in \mathbf{K}$, the \mathbb{C} -algebras A_λ and B_λ are said to be *deformations* of A and B respectively.

Lemma 2.3. If $b \in B$ is a central element then $\bar{b} \in B_1$ is a Poisson central element.

Proof. It is clear by (2.3). \square

In the following theorem, note that the maps α_1, δ_1 are constructed in a different way from $\alpha_\lambda, \delta_\lambda$ for $\lambda \in \mathbf{K}$.

Theorem 2.4. [8, Proposition 4.1] $B_1 \cong A_1[z; \alpha_1, \delta_1]_p$ as Poisson \mathbb{C} -algebras, where α_1 and δ_1 are defined by

$$\begin{aligned} \alpha_1(\bar{a}) &= \overline{(t - 1)^{-1}(\alpha(a) - a)}, \\ \delta_1(\bar{a}) &= \overline{(t - 1)^{-1}\delta(a)} \end{aligned}$$

for all $a \in A$.

For each $\lambda \in \mathbf{K}$, let $\mathbb{C}_\lambda = \mathbb{C}$. Note that, for $f(t) \in \mathbb{F}$, the complex number $f(\lambda)$ is well-defined by (2.1). Define a \mathbb{C} -algebra homomorphism

$$\gamma_{\mathbb{F}} : \mathbb{F} \longrightarrow \prod_{\lambda \in \mathbf{K}} \mathbb{C}_\lambda, \quad \gamma_{\mathbb{F}}(f(t)) = (f(\lambda))_{\lambda \in \mathbf{K}}.$$

Lemma 2.5. *The \mathbb{C} -algebra homomorphism $\gamma_{\mathbb{F}}$ is injective.*

Proof. Suppose that $\gamma_{\mathbb{F}}(f(t)) = 0$. Then $f(\lambda) = 0$ for all $\lambda \in \mathbf{K}$. Since \mathbf{K} is an infinite set and every nonzero polynomial has only finite zeros, $f(t) = 0$. Hence $\gamma_{\mathbb{F}}$ is injective. \square

Set

$$\widehat{A} = \prod_{\lambda \in \mathbf{K}} A_\lambda, \quad \widehat{B} = \prod_{\lambda \in \mathbf{K}} B_\lambda.$$

Let

$$\pi_\lambda : \widehat{A} \longrightarrow A_\lambda, \quad \pi_\lambda : \widehat{B} \longrightarrow B_\lambda$$

be the canonical projections onto A_λ and B_λ for each $\lambda \in \mathbf{K}$ and let γ_A and γ_B be the \mathbb{C} -algebra homomorphisms

$$(2.4) \quad \begin{aligned} \gamma_A : A &\longrightarrow \widehat{A}, & \gamma(a) &= (\overline{a})_{\lambda \in \mathbf{K}}, \\ \gamma_B : B &\longrightarrow \widehat{B}, & \gamma(b) &= (\overline{b})_{\lambda \in \mathbf{K}}. \end{aligned}$$

Thus $\pi_\lambda \gamma_A(a) = \overline{a}$ and $\pi_\lambda \gamma_B(b) = \overline{b}$ for each $\lambda \in \mathbf{K}$.

Lemma 2.6. *γ_A is injective if and only if γ_B is injective.*

Proof. If γ_B is injective then γ_A is also injective since γ_A is the restriction of γ_B to A . Suppose that γ_A is injective and that $\gamma_B(\sum_i a_i z^i) = 0$, where $a_i \in A$. Then $\overline{a_i} = 0$ for each $\lambda \in \mathbf{K}$. It follows $\gamma_A(a_i) = 0$. Hence $a_i = 0$ for all i since $\gamma_A(a_i) = 0$ and γ_A is injective. \square

Theorem 2.7. *Suppose that γ_A is injective. Set*

$$\begin{aligned} \Gamma_A : \gamma_A(A) &\longrightarrow A_1, & \Gamma_A(x) &= \overline{\gamma_A^{-1}(x)}, \\ \Gamma_B : \gamma_B(B) &\longrightarrow B_1, & \Gamma_B(x) &= \overline{\gamma_B^{-1}(x)}. \end{aligned}$$

Then the \mathbb{C} -algebra homomorphisms Γ_A and Γ_B are surjective.

Proof. By Lemma 2.6, γ_A and γ_B are injective and thus Γ_A and Γ_B are well-defined \mathbb{C} -algebra homomorphisms. Since the canonical maps from A into A_1 is surjective, Γ_A is surjective clearly. Similarly Γ_B is surjective. \square

3. POISSON GENERALIZED WEYL ALGEBRA

The following quantum generalized Weyl algebra $A(a(h), q)$ is a special case of the generalized Weyl algebra introduced by Bavula in [1].

Definition 3.1. Let $0 \neq q \in \mathbb{C}$ be not a root of unity and let $0 \neq a(h) \in \mathbb{C}[h^{\pm 1}]$. The quantum generalized Weyl algebra $A(a(h), q)$ is the \mathbb{C} -algebra generated by $h^{\pm 1}, x, y$ subject to the relations

$$xh = qhx, \quad yh = q^{-1}hy, \quad xy = a(qh), \quad yx = a(h), \quad h^{\pm 1}h^{\mp 1} = 1.$$

Set

$$\mathbb{F} = \mathbb{C}[t, t^{-1}], \quad \mathbf{K} = \mathbb{C} \setminus (\{0, 1\} \cup \{\text{roots of unity}\}).$$

Assume throughout the section that $0 \neq q \in \mathbb{C}$ is **not a root of unity** and that $0 \neq a(h) \in \mathbb{C}[h^{\pm 1}]$ is **not invertible**. Hence $q \in \mathbf{K}$ and $a(h)$ has at least two nonzero terms.

Set

$$B := \mathbb{F}[h^{\pm 1}][x; \alpha][y; \beta, \delta],$$

where

$$(3.1) \quad \begin{aligned} \alpha(h) &= th, \\ \beta(h) &= t^{-1}h, \quad \beta(x) = x, \\ \delta(h) &= 0, \quad \delta(x) = a(h) - a(th). \end{aligned}$$

Lemma 3.2. *The element $xy - a(th) \in B$ is a central element.*

Proof. It is proved routinely by (3.1). □

Denote by $B(a(h), q)$ the \mathbb{C} -algebra generated by $h^{\pm 1}, x, y$ subject to the relations

$$(3.2) \quad xh = qhx, \quad yh = q^{-1}hy, \quad yx = xy + a(h) - a(qh), \quad h^{\pm 1}h^{\mp 1} = 1,$$

which is obtained from B by substituting q for t . The \mathbb{C} -algebra $B(a(h), q)$ is an iterated skew polynomial algebra

$$B(a(h), q) = \mathbb{C}[h^{\pm 1}][x; \alpha_q][y; \beta_q, \delta_q],$$

where $\alpha_q, \beta_q, \delta_q$ are the maps induced by α, β, δ respectively. Moreover

$$(3.3) \quad B(a(h), q) \cong B/(t - q)B = B_q$$

as \mathbb{C} -algebras by Lemma 2.2(2). For each $\lambda \in \mathbf{K}$, λ is not a root of unity and thus there exists the \mathbb{C} -algebra $B(a(h), \lambda) \cong B_\lambda$ which is obtained from (3.3) by substituting λ for q . Observe that q is not only a nonzero and non-root of unity but also plays a role as a parameter taking values in \mathbf{K} .

Observation 3.3. In $B(a(h), q)$, q plays a role as a parameter taking values in \mathbf{K} and thus, for each $\lambda \in \mathbf{K}$, there exists an evaluation map

$$(3.4) \quad e_\lambda : B(a(h), q) \longrightarrow B_\lambda, \quad f(q) \mapsto f(\lambda).$$

Note that the 5-tuples $(\mathbf{K}, \mathbb{F}, \mathbb{F}[h^{\pm 1}], t - 1, (\alpha, 0))$ and $(\mathbf{K}, \mathbb{F}, \mathbb{F}[h^{\pm 1}][x; \alpha], t - 1, (\beta, \delta))$ satisfy Notation 2.1(1)-(5). Applying Theorem 2.4 to $\mathbb{F}[h^{\pm 1}][x; \alpha]$ and B , there exists the Poisson \mathbb{C} -algebra

$$B_1 = \mathbb{C}[h^{\pm 1}][x; \alpha_1]_p[y; \beta_1, \delta_1]_p,$$

where

$$(3.5) \quad \begin{aligned} \alpha_1(h) &= h, \\ \beta_1(h) &= -h, \quad \beta_1(x) = 0, \\ \delta_1(h) &= 0, \quad \delta_1(x) = -a'(h)h, \end{aligned}$$

here $a'(h)$ is the formal derivative of $a(h)$.

Define a \mathbb{C} -algebra homomorphism γ_B by

$$\gamma_B : B \longrightarrow \widehat{B} = \prod_{\lambda \in \mathbf{K}} B_\lambda, \quad \gamma_B(z) = (\overline{z})_{\lambda \in \mathbf{K}}.$$

Applying Lemma 2.6 to γ_B inductively, γ_B is injective by Lemma 2.5 and thus, by Theorem 2.7, there exists the \mathbb{C} -algebra homomorphism

$$\Gamma_B : \gamma_B(B) \longrightarrow B_1, \quad \Gamma_B = \gamma_1 \gamma_B^{-1},$$

where $\gamma_1 : B \longrightarrow B_1 = B/(t-1)B$ is the canonical projection.

Define a map (not \mathbb{C} -linear map)

$$\hat{\cdot} : B(a(h), q) \longrightarrow \hat{B} = \prod_{\lambda \in \mathbf{K}} B_\lambda, \quad f(q) \mapsto \widehat{f(q)} := (f(\lambda))_{\lambda \in \mathbf{K}}$$

which exists by (3.4). Note that the image of $\hat{\cdot}$ is equal to the image of γ_B . Hence there exists the composition $\Gamma := \Gamma_B \circ \hat{\cdot}$. Roughly speaking, Γ is a map defined by plugging 1 to q .

$$(3.6) \quad \Gamma : B(a(h), q) \longrightarrow \hat{B} = \prod_{\lambda \in \mathbf{K}} B_\lambda \longleftarrow B \longrightarrow B_1.$$

Note that Γ is surjective. By Lemma 2.3 and Lemma 3.2, $\Gamma(xy - a(qh)) = xy - a(h)$ is a Poisson central element of B_1 . Thus $(xy - a(h))B_1$ is a Poisson ideal. Set

$$A_1 := B_1/(xy - a(h))B_1.$$

Henceforth, we simply write w for $\overline{w} \in A_1$.

Theorem 3.4. (1) The quantum generalized Weyl algebra $A(a(h), q)$ is equal to the factored \mathbb{C} -algebra $B(a(h), q)/(xy - a(qh))B(a(h), q)$.

(2) The Poisson algebra A_1 is induced from $A(a(h), q)$ by Γ in (3.6).

(3) The Poisson algebra A_1 is the \mathbb{C} -algebra $\mathbb{C}[h^{\pm 1}, x, y]/\langle xy - a(h) \rangle$ with Poisson bracket

$$(3.7) \quad \{x, h\} = hx, \quad \{y, h\} = -hy, \quad \{y, x\} = -a'(h)h.$$

We will call A_1 a Poisson generalized Weyl algebra.

Proof. (1) It follows by (3.2).

(2) It follows by the facts that $\Gamma(B(a(h), q)) = B_1$ and $\Gamma(xy - a(qh)) = xy - a(h)$.

(3) Since $B_1 = \mathbb{C}[h^{\pm 1}, x, y]$, the result follows by (3.5). \square

Let R be a Poisson algebra. If there exists subspaces R_k , $k \in \mathbb{Z}$, such that $R = \bigoplus_{k \in \mathbb{Z}} R_k$ and

$$R_k R_\ell \subseteq R_{k+\ell}, \quad \{R_k, R_\ell\} \subseteq R_{k+\ell}$$

for all $k, \ell \in \mathbb{Z}$ then R is said to be a \mathbb{Z} -graded Poisson algebra.

Give degrees on the generators h, x, y of B_1 by $\deg(h) = 0$, $\deg(x) = 1$ and $\deg(y) = -1$. Then B_1 is a \mathbb{Z} -graded Poisson algebra. Moreover the Poisson ideal $\langle xy - a(h) \rangle$ is graded. Thus A_1 is also a \mathbb{Z} -graded Poisson algebra

$$(3.8) \quad A_1 = \bigoplus_{k \in \mathbb{Z}} W_k,$$

where

$$W_k = \begin{cases} \mathbb{C}[h^{\pm 1}]x^k & \text{if } k > 0, \\ \mathbb{C}[h^{\pm 1}] & \text{if } k = 0, \\ \mathbb{C}[h^{\pm 1}]y^{-k} & \text{if } k < 0. \end{cases}$$

Lemma 3.5. *Define a \mathbb{C} -linear map f by*

$$f : A_1 \longrightarrow A_1, \quad f(a) = \{a, h\}h^{-1}.$$

Then, for each $k \in \mathbb{Z}$, every nonzero element of W_k is an eigenvector of f with eigenvalue k .

Note that $\mathbb{C}[h^{\pm 1}]$ is a unique factorization domain since it is a principal ideal domain.

Theorem 3.6. *The Poisson algebra A_1 is Poisson simple if and only if every root of $a(h)$ is a simple root.*

Proof. Let $b(h) \in \mathbb{C}[h^{\pm 1}]$ be the greatest common divisor of $a(h)$ and $a'(h)$. Note that $a(h)$ has a root with multiplicity > 1 if and only if $b(h)$ is a nonunit.

(\Rightarrow) Suppose that $a(h)$ has a root with multiplicity > 1 . Thus $b(h)$ is a nonunit. Let I be the ideal of A_1 generated by x, y and $b(h)$. Then A_1/I is isomorphic to the algebra $\mathbb{C}[h^{\pm 1}, x, y]/J$, where J is the ideal of $\mathbb{C}[h^{\pm 1}, x, y]$ generated by x, y and $b(h)$, since $a(h)$ is divided by $b(h)$. Moreover $\mathbb{C}[h^{\pm 1}, x, y]/J$ is isomorphic to $\mathbb{C}[h^{\pm 1}]/b(h)\mathbb{C}[h^{\pm 1}]$. Thus I is a nontrivial ideal of A_1 . Observe that

$$\begin{aligned} \{x, b(h)\} &= b'(h)\{x, h\} = b'(h)a'(h)hx \in I, \\ \{y, b(h)\} &= b'(h)\{y, h\} = -b'(h)a'(h)hy \in I \\ \{y, x\} &= -a'(h)h \in I \end{aligned}$$

by the chain rule and (3.7). Hence I is a nontrivial Poisson ideal of A_1 and thus A_1 is not Poisson simple.

(\Leftarrow) Suppose that A_1 is not Poisson simple. Then there exists a nontrivial Poisson ideal I of A_1 . Let P be a minimal prime ideal over I . By [4, 6.2], P is a prime Poisson ideal. Applying f to P , P contains a nonzero element $w_k \in W_k$ for some $k \in \mathbb{Z}$ by Lemma 3.5. If $k \geq 0$ then $x \in P$ or $c(h) \in P$ for some $0 \neq c(h) \in \mathbb{C}[h^{\pm 1}]$ since P is prime. If $x \in P$ then $a'(h)h = \{x, y\} \in P$ and thus $a(h) \in P$ and $a'(h) \in P$. It follows that the greatest common divisor $b(h)$ of $a(h)$ and $a'(h)$ is an element of P and thus P contains a unit, a contradiction. Similarly, if $k < 0$ then repeating the argument gives that $y \notin P$ and that P contains an element $0 \neq c(h) \in \mathbb{C}[h^{\pm 1}]$.

Let $0 \neq c(h) \in P$ and $x, y \notin P$. We may assume $c(h) \in \mathbb{C}[h]$ by multiplying h^n to $c(h)$ for sufficiently large n . If the degree of $c(h)$ is greater than 1 then $0 \neq c'(h) \in P$ since

$$P \ni \{x, c(h)\} = c'(h)hx.$$

Repeating this process, we get that P contains a unit, a contradiction. Hence A_1 is Poisson simple. \square

Let us find the Poisson endomorphisms of A_1 . The following arguments are Poisson analogues of those in [7]. For completion, we repeat them. Note that the unit group of A_1 is $\{\gamma h^i \mid \gamma \in \mathbb{C}^*, i \in \mathbb{Z}\}$. Let ψ be a Poisson endomorphism of A_1 . Then $\psi(h) = \gamma h^i$ for some $\gamma \in \mathbb{C}^*$ and $i \in \mathbb{Z}$ since Poisson endomorphisms preserve units. Hence there are possible three types of Poisson endomorphisms of A_1 as in [7]. Positive-type Poisson endomorphisms, that is, Poisson endomorphisms ψ such that $\psi(h) = \gamma h^i$ ($i > 0$); Zero-type Poisson endomorphisms, that is, Poisson endomorphisms ψ such that $\psi(h) = \gamma$; Negative-type Poisson endomorphisms, that is, Poisson endomorphisms ψ such that $\psi(h) = \gamma h^i$ ($i < 0$).

In the following theorem, we see that the positive-type Poisson endomorphisms and the negative-type Poisson endomorphisms of A_1 are the same forms as those of endomorphisms of $A(a(h), q)$ but the zero-type Poisson endomorphisms of A_1 are slightly different from those

of $A(a(h), q)$. (One should compare the following theorem with [7, Proposition 3.1, Proposition 4.1 and Proposition 5.3].)

Theorem 3.7. *Let d be the maximal degree of $a(h)$. Write $a(h)$ by*

$$a(h) = a_{i_1}h^{i_1} + a_{i_2}h^{i_2} + \dots + a_{i_m}h^{i_m},$$

where $i_1 < i_2 < \dots < i_m = d$ and $a_{i_j} \in \mathbb{C}^$ for all $j = 1, 2, \dots, m$. Denote by k the greatest common divisor of $d - i_1, d - i_2, \dots, d - i_{m-1}$.*

(1) Let ψ be a positive-type Poisson endomorphism of A_1 . Then

$$(3.9) \quad \psi(h) = \gamma h, \quad \psi(x) = bh^n x, \quad \psi(y) = \gamma^d b^{-1} h^{-n} y,$$

where γ is a k -th root of unity, $b \in \mathbb{C}^$ and $n \in \mathbb{Z}$. Conversely, a map ψ satisfying (3.9) determines a unique Poisson automorphism of A_1 .*

(2) Let ψ be a zero-type Poisson endomorphism of A_1 . Then

$$(3.10) \quad \psi(h) = \gamma, \quad \psi(x) = 0, \quad \psi(y) = 0,$$

where $\gamma \in \mathbb{C}^$ is a root of $a(h)$ with multiplicity > 1 . If $a(h)$ has a root with multiplicity > 1 then a map ψ satisfying (3.10) determines a unique Poisson endomorphism. If every root of $a(h)$ has multiplicity 1 then there are no zero-type Poisson endomorphisms.*

(3) Let ψ be a negative-type Poisson endomorphism of A_1 . Then

$$(3.11) \quad \psi(h) = \gamma h^{-1}, \quad \psi(x) = ch^v y, \quad \psi(y) = bh^u x,$$

where $\gamma, b, c \in \mathbb{C}^$ and $u, v \in \mathbb{Z}$ satisfy the relation*

$$(3.12) \quad bch^{u+v}a(h) = a(\gamma h^{-1}).$$

Conversely, a map ψ satisfying (3.11) determines a unique Poisson automorphism of A_1 .

Proof. Note that every Poisson endomorphism ψ of A_1 preserves the following equations

$$(3.13) \quad \{x, h\} = hx, \quad \{y, h\} = -hy, \quad \{y, x\} = -a'(h)h, \quad xy = a(h).$$

(1) Let ψ be a Poisson endomorphism such that $\psi(h) = \gamma h^j$ for $\gamma \in \mathbb{C}^*$ and $j > 0$. Suppose that $\psi(x) = 0$. Applying ψ to the third equation of (3.13), the left hand side is zero and the right hand side is $-\gamma a'(\gamma h^j)h^j$ which is nonzero, a contradiction. Hence $\psi(x) \neq 0$. Similarly $\psi(y) \neq 0$. We can set $\psi(x) = \sum_{k \in \mathbb{Z}} w_k$ by (3.8), where $w_k \in W_k$ for each k . Applying ψ to $\{x, h\} = hx$, we have

$$\{\psi(x), \gamma h^j\} = \gamma h^j \psi(x).$$

The left hand side of the above equation is $\sum_k jk \gamma h^j w_k$ and thus $jk = 1$ for all k such that $w_k \neq 0$. Thus we have that

$$\psi(h) = \gamma h, \quad \psi(x) = b(h)x$$

for some $0 \neq b(h) \in \mathbb{C}[h^{\pm 1}]$. Repeating this argument on $\{y, h\} = -hy$, we get

$$\psi(y) = c(h)y$$

for some $0 \neq c(h) \in \mathbb{C}[h^{\pm 1}]$.

Applying ψ to the last equation of (3.13), we get $b(h)c(h)a(h) = a(\gamma h)$. Comparing the maximal and the minimal degrees for h on both sides and then coefficients, we get

$$\psi(x) = bh^n x, \quad \psi(y) = \gamma^d b^{-1} h^{-n} y,$$

where γ is a k -th root of unity, $b \in \mathbb{C}^*$ and $n \in \mathbb{Z}$.

Conversely, let ψ be a map satisfying (3.9). Then ψ determines a unique algebra endomorphism since B_1 is the commutative polynomial ring $\mathbb{C}[h^{\pm 1}, x, y]$ and it preserves the last equation of (3.13). It is checked routinely that ψ preserves the other equations of (3.13). Thus ψ is a Poisson endomorphism. Such ψ is a Poisson automorphism since there exists ψ^{-1} defined by

$$\psi^{-1}(h) = \gamma^{-1}h, \quad \psi^{-1}(x) = \gamma^n b^{-1} h^{-n} x, \quad \psi^{-1}(y) = \gamma^{-n-d} b h^n y.$$

(2) Let ψ be a Poisson endomorphism such that $\psi(h) = \gamma$, where $\gamma \in \mathbb{C}^*$. Then $\psi(x) = \psi(y) = 0$ by the first and the second equations of (3.13). Applying ψ to the third and the last equations of (3.13), we get that γ is a common root of $a(h)$ and $a'(h)$. Thus γ is a root of $a(h)$ with multiplicity > 1 .

Conversely, let ψ be a map satisfying (3.10). Then ψ determines a unique algebra endomorphism since ψ preserves the last equation of (3.13). Moreover ψ preserves the other equations of (3.13) and thus ψ is a Poisson endomorphism. Now the other statements are trivial by Theorem 3.6.

(3) Let ψ be a Poisson endomorphism such that $\psi(h) = \gamma h^j$ for $\gamma \in \mathbb{C}^*$ and $j < 0$. Since ψ^2 is of positive type, $\psi(x) \neq 0$ and $\psi(y) \neq 0$ by (1). We can set $\psi(x) = \sum_{k \in \mathbb{Z}} w_k$ by (3.8), where $w_k \in W_k$ for each k . Applying ψ to $\{x, h\} = hx$, we have

$$\{\psi(x), \gamma h^j\} = \gamma h^j \psi(x).$$

The left hand side of the above equation is $\sum_k jk \gamma h^j w_k$ and thus $jk = 1$ for all k such that $w_k \neq 0$. Thus we have that

$$\psi(h) = \gamma h^{-1}, \quad \psi(x) = c(h)y$$

for some $0 \neq c(h) \in \mathbb{C}[h^{\pm 1}]$. Repeating this argument on $\{y, h\} = -hy$, we get

$$\psi(y) = b(h)x$$

for some $0 \neq b(h) \in \mathbb{C}[h^{\pm 1}]$. Moreover we have that $b(h) = bh^u, c(h) = ch^v$ for some $b, c \in \mathbb{C}^*$ and $u, v \in \mathbb{Z}$ by (1) since ψ^2 is of positive type. Applying ψ to the last equation of (3.13), we get the relation $bch^{u+v}a(h) = a(\gamma h^{-1})$.

Conversely, let ψ be a map satisfying (3.11). Then ψ preserves the last equation of (3.13) and thus ψ determines a unique algebra endomorphism. It is easy to check that ψ preserves the first and the second equations of (3.13). Moreover it is shown that ψ preserves the third equation of (3.13) by differentiating the equation (3.12) by h . Hence ψ satisfying (3.11) determines a unique Poisson endomorphism. Since there exists a map ψ^{-1} of the negative type, ψ determines a unique Poisson automorphism. \square

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